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The stability of laminar boundary layers at separation

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The effect of an adverse pressure gradient on the stability of a laminar boundary layer is considered in the limiting case when the skin friction at the wall vanishes, i.e. when $U'(0) = 0$. Such flows are not absolutely unstable as might have been expected but have a minimum critical Reynolds number of the order of 25. General results are given for the asymptotic behaviour of both the upper and lower branches of the neutral curve and a complete neutral curve is obtained for Pohlhausen's simple fourth-degree polynomial profile at separation.

1. Introduction

The effect of a pressure gradient on the stability of laminar boundary layers has been widely studied and the destabilizing effect of an adverse pressure gradient is well known. In none of the existing calculations, however, has the limiting case of a boundary layer *at separation* been considered.‡ This limiting case would appear to be unusual for at least two reasons: first, because there is a substantial change in the structure of the characteristic equation when

$$U'(0) = 0$$

and, secondly, because the simple formulas given by Lin (1955, p. 83) for estimating minimum critical Reynolds numbers clearly fail in such a case. In fact, these formulas would suggest that the minimum critical Reynolds number for such flows is actually zero and hence that they are absolutely unstable. The present calculations were undertaken, therefore, to examine just this possibility. Although detailed calculations have been made for only one particular velocity profile, the results are likely to be typical of all boundary-layer profiles that are monotonically increasing with $U'(0) = 0$. Such profiles necessarily have an inflexion point in the interval $0 < y < \infty$ and are therefore unstable in the inviscid limit.

Perhaps the most realistic basic flow for the present purposes would have been Thwaites's (1949) little-known but exact solution of the Falkner–Skan equation for $\beta = -1$. The analytical form of this profile is sufficiently complicated, however, to render it somewhat unsuitable for the purposes of a stability calculation. Since the primary purpose of the present calculation was merely to illustrate the essential stability characteristics of boundary layers at separation, it was felt that even a crude approximation to the velocity profile would be entirely

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‡ Except for an incomplete calculation by Pretsch (1941) based on the Falkner–Skan velocity profile at separation.

adequate, and it was decided therefore to use the simple Pohlhausen fourth-degree polynomial profile. At separation the parameter Λ of the Pohlhausen P4 profiles has the value -12 and the velocity distribution is then given by (see, for example, Goldstein 1938)

$$U(y) = \begin{cases} y^2(6-8y+3y^2) & \text{for} \quad 0 \leqslant y \leqslant 1, \\ 1 & \text{for} \quad 1 \leqslant y \leqslant \infty. \end{cases} \tag{1.1}$$

In the following sections, therefore, we shall consider the solution of the Orr–Sommerfeld equation within the framework of the usual asymptotic approximations for the velocity distribution (1.1). In a more realistic treatment of this problem, however, one would have to allow not only for the non-parallel character of the basic flow near separation but also for the stream-wise variation of both components of the basic velocity.

2. The solution of the inviscid equation

In the solution of the Orr–Sommerfeld equation for flows of the boundary-layer type it is necessary to obtain approximations to the two solutions that remain bounded as $y \to +\infty$. One of the required approximations has an essentially inviscid character, being merely the solution of the inviscid equation

$$(U-c)(\phi''-\alpha^2\phi)-U''\phi = 0 \tag{2.1}$$

that remains bounded as $y \to +\infty$. We shall denote such a solution of equation (2.1) by $\Phi(y)$ and, for convenience, normalize it so that $\Phi(y_c) = 1$. For the velocity profile (1.1), the relevant 'outer' solution of equation (2.1) is simply

$$\Phi(y) = Ke^{-\alpha y} \quad \text{for} \quad 1 \leqslant y \leqslant \infty, \tag{2.2}$$

where the constant K may depend on both α and c.

On the interval $0 \leqslant y \leqslant 1$, $\Phi(y)$ can most conveniently be expressed as a linear combination of the Tollmien solutions

$$\phi_A(y) = (y-y_c)P_A(y-y_c) \tag{2.3}$$

and

$$\phi_B(y) = P_B(y-y_c)+(U_c''/U_c')\,\phi_A(y)\log(y-y_c), \tag{2.4}$$

where P_A and P_B are power series in $y-y_c$, the leading coefficients of which are unity, and, to be definite, we suppose that ϕ_B contains no multiple of ϕ_A. Thus, we have

$$\Phi(y) = A\phi_A(y)+\phi_B(y), \tag{2.5}$$

where the constant A may also depend on both α and c. The requirement that Φ and Φ' be continuous at $y = 1$ leads to two inhomogeneous equations for the constants A and K which can be solved to give

$$A = -\frac{\alpha\phi_B(1)+\phi_B'(1)}{\alpha\phi_A(1)+\phi_A'(1)} \quad \text{and} \quad K = \frac{e^{\alpha}}{\alpha\phi_A(1)+\phi_A'(1)}, \tag{2.6}$$

where we have used the fact that the Wronskian $W(\phi_A, \phi_B) = -1$. The solution (2.5) has a logarithmic branch point at $y = y_c$ and, as a more detailed investigation shows, it provides a valid asymptotic approximation only in the sector $-\frac{7}{6}\pi < \arg(y-y_c) < \frac{1}{6}\pi$ of the complex y-plane excluding the immediate neighbourhood of y_c. For the present purposes, however, it is not necessary to consider the viscous corrections to ϕ_B.

In the actual computation of $\Phi(y)$, $\phi_A(y)$ and the regular part of $\phi_B(y)$ were obtained by numerical integration and the power series $P_A(y-y_c)$ and $P_B(y-y_c)$ serve only to provide the necessary initial data. This procedure is identical with the one described by Hughes & Reid (1965; hereinafter referred to as I) for the asymptotic suction profile and need not be described further here.

3. The solution of the characteristic equation

An approximation to the second bounded solution of the Orr–Sommerfeld equation can be obtained from the equation $\phi^{iv} = i\alpha R U_c'(y-y_c)\,\phi''$, and the solution of this equation that remains bounded as $y \to +\infty$ can be written in the form

$$\phi_3(y) = \int_{\infty_1}^{\xi} d\xi \int_{\infty_1}^{\xi} \mathrm{Ai}(\xi)\, d\xi, \tag{3.1}$$

where $\xi = (y-y_c)/\epsilon$, $\epsilon = (i\alpha R U_c')^{-\frac{1}{3}}$, and ∞_1 denotes a path of integration that tends to infinity in the sector $|\arg \xi| < \frac{1}{3}\pi$, and so makes a negligible contribution on the interval $1 \leqslant y \leqslant \infty$.

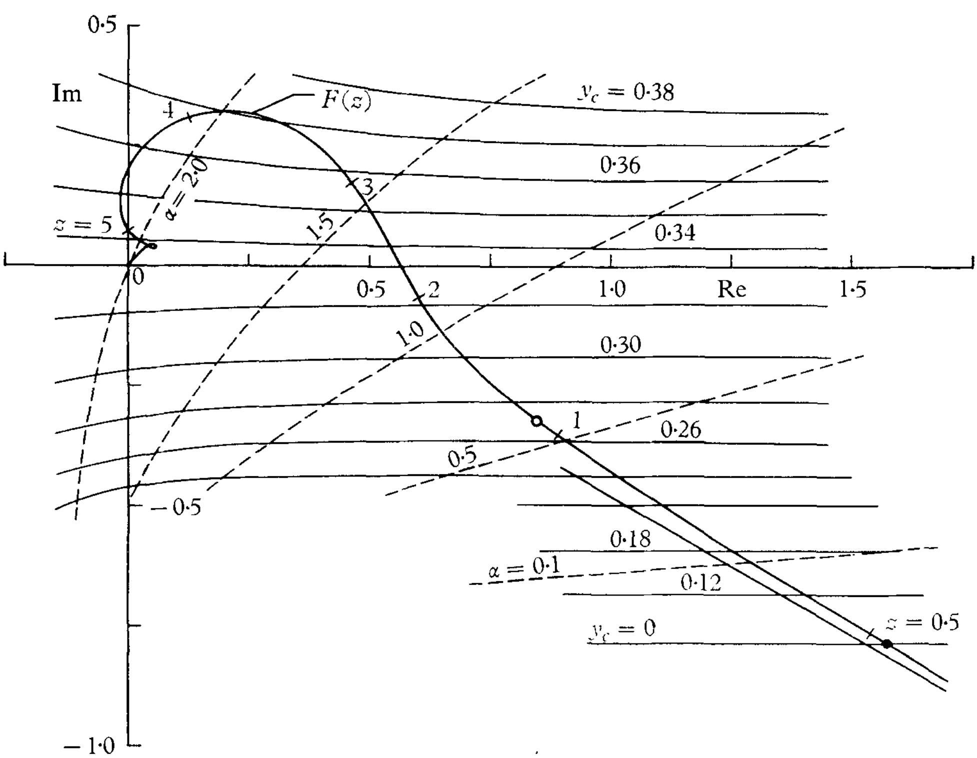

FIGURE 1. The graphical solution of the characteristic equation (3.2) for the velocity profile (1.1). The open circle corresponds to the minimum critical Reynolds number and the solid circle corresponds to the asymptote to the lower branch of the neutral curve. The Tietjens function is from Miles (1960).

The required approximation to the general solution of the Orr–Sommerfeld equation is thus a linear combination of $\Phi(y)$ and $\phi_3(y)$. When the boundary conditions $\phi(0) = \phi'(0) = 0$ are applied to this solution, the characteristic equation is found to be of the form

$$E(\alpha, c) = F(z), \quad \text{where} \quad E(\alpha, c) = -\Phi(0)/y_c \Phi'(0) \tag{3.2}$$

and $F(z)$ is the Tietjens function with argument $z = (\alpha R U_c')^{\frac{1}{3}} y_c$. This result can also be obtained by allowing $U'(0) \to 0$ in the usual form of the characteristic equation [cf. I, equations (4.1) and (4.2)]. This characteristic equation was then solved by the usual graphical procedure in which the real and imaginary parts of both sides of equation (3.2) are plotted on the same graph as shown in figure 1. In evaluating the inviscid part of the characteristic equation, it was found to be more convenient to evaluate $E(\alpha, c)$ for assigned values of y_c rather than for assigned values of c as is usually done.

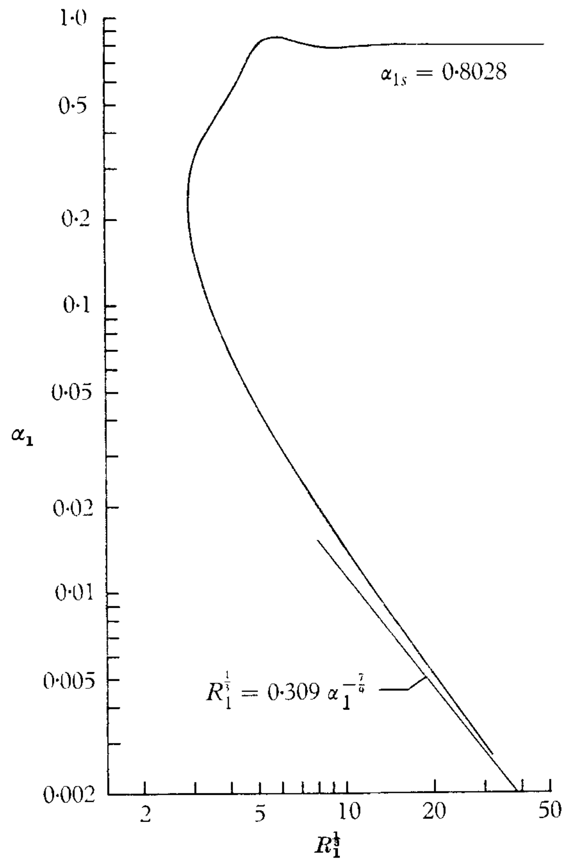

FIGURE 2. The curve of neutral stability for the velocity profile (1.1).

For this separating profile it was found necessary to use values of $F(z)$ that lie below the real axis and correspond to $z < z_0\,(= 2{\cdot}297)$. In fact, the point where R reaches its minimum value and the point corresponding to the asymptote to the lower branch of the neutral curve both fall below the real axis. In spite of these somewhat unusual features of the solution, the curve of neutral stability shown in figure 2 is essentially similar to the neutral curves found for boundary-layer flows having an inflexion point but with $U'(0) > 0$. The relationship between α and c along the neutral curve is shown in figure 3.

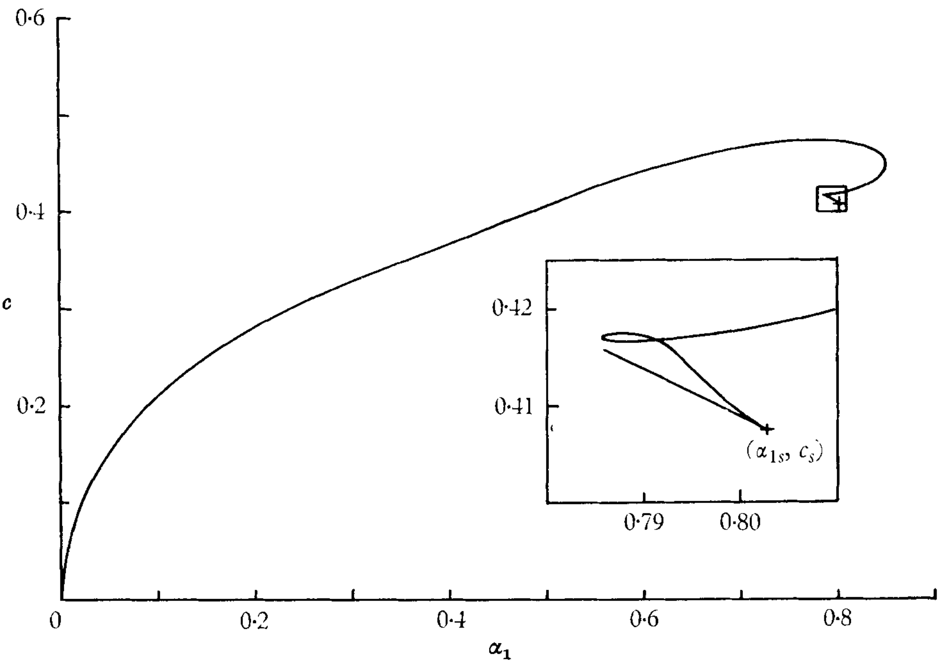

FIGURE 3. The relationship between the wave-number α_1 and the wave-speed c along the neutral curve.

To facilitate the comparison of these results with those for other boundary-layer profiles, we have changed the characteristic length scale from L_*, which has only been defined implicitly by equation (1.1), to the displacement thickness δ_1. The wave-number α_1 and Reynolds number R_1 (based on δ_1) are then related to α and R (based on L_*) by

$$\alpha_1 = \tfrac{2}{5}\alpha \quad \text{and} \quad R_1 = \tfrac{2}{5}R. \tag{3.3}$$

The loop in the Tietjens function shown in figure 1 causes the small loop in the (α, c)-curve and also causes the concave bend along the upper branch of the neutral curve just before that branch tends to its limiting value. These peculiarities of the solution would not appear to have any physical significance but are most likely due to a slight inadequacy in the inviscid approximation $\Phi(y)$.

4. The asymptotes to the curve of neutral stability

To complete the discussion of the neutral curve for this problem it is necessary to obtain its asymptotic behaviour as $R \to \infty$. Since the limiting behaviour of the upper and lower branches are quite different, the two limits must be treated separately. Along the upper branch we approach a purely inviscid limit (as would be expected for a profile having an inflexion point) and this limit does not depend in any essential way on the fact that $U'(0) = 0$. Along the lower branch, however, the critical point approaches the boundary as $R \to \infty$ and viscous effects never become negligible. The asymptote to this branch of the neutral curve emerges in a somewhat unusual way and does depend critically on the vanishing of $U'(0)$.

The limiting inviscid solution

To determine the asymptotic behaviour of the upper branch of the neutral curve it is first necessary to obtain the limiting inviscid solution. According to the usual purely inviscid analysis of the stability of boundary-layer flows (see, for example, Lin 1955, §8·2), if the velocity profile has an inflexion point, then

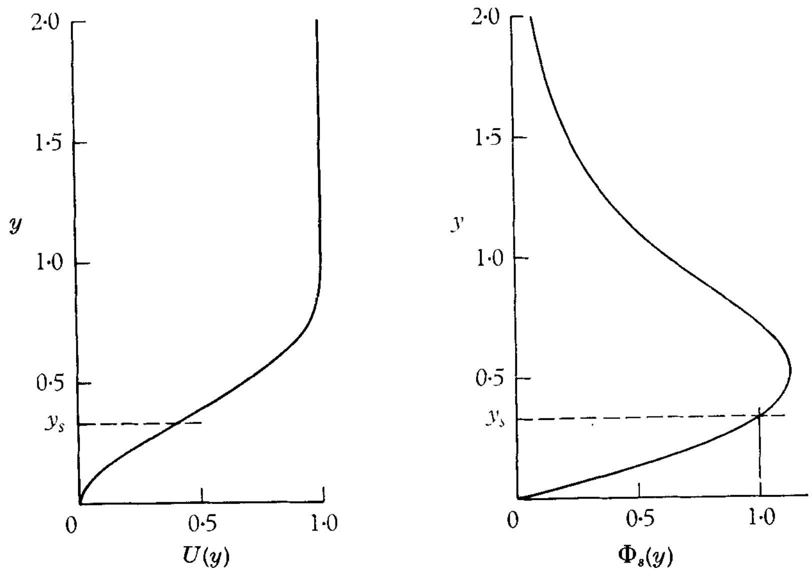

FIGURE 4. The velocity profile (left) and the limiting inviscid solution (right).

the flow is unstable for a non-zero range of the wave-number α. This range of α is bounded by the neutral state $\alpha = \alpha_s > 0$ and $c_s = U(y_s)$, where $U''(y_s) = 0$ and $c_i = 0$. As $R \to \infty$ along the upper branch of the neutral curve, the limiting inviscid solution must emerge, therefore, as the solution of the inviscid equation that (i) vanishes at $y = 0$, (ii) is regular at the critical point, and (iii) remains bounded as $y \to +\infty$.

The velocity profile (1.1) has an inflexion point at $y_s = \frac{1}{3}$ and at that point $c_s = U(y_s) = \frac{11}{27}$. The condition (iii) above is automatically satisfied by the matching conditions (2.6) and the condition (ii) is satisfied provided $U''(y_c) = 0$, i.e. provided $y_c = y_s = \frac{1}{3}$ and $c = c_s = \frac{11}{27}$. The required inviscid solution is thus one of the family of solutions given by

$$\Phi(y) = \begin{cases} A\phi_A(y) + P_B(y-y_s) & \text{for} \quad 0 \leqslant y \leqslant 1 \\ Ke^{-\alpha y} & \text{for} \quad 1 \leqslant y \leqslant \infty, \end{cases} \tag{4.1}$$

where $P_B(y-y_s)$ is the regular part of $\phi_B(y)$. Finally, α_s must be chosen to satisfy the condition (i). By computing $\Phi(0)$ for a range of values of α and then interpolating for α_s, it was found that the parameters of the solution must have the values

$$\alpha_s = 2{\cdot}00708, \quad A = 1{\cdot}5358, \quad K = 4{\cdot}6595, \quad \Phi_s'(0) = 4{\cdot}0051. \tag{4.2}$$

The limiting inviscid solution $\Phi_s(y)$ found in this way is shown in figure 4.

It is interesting to note, however, that this inviscid limit also emerges in a completely natural way from the full viscous calculation described in §3. For as $z \to +\infty$, $F(z) \to 0$ and we must therefore find the values of α and c for which $E(\alpha, c) = 0$. This can easily be done by finding the lines α = constant and c = constant in figure 1 that pass through the origin. The required c-line is simply the real axis and corresponds to $y_c = y_s = \frac{1}{3}$. For velocity profiles having more than one critical point, the required eigenfunction in the inviscid limit is no longer regular and, since the value of c_s is not known *a priori*, this latter method of solution may be useful in such cases.

The asymptotic behaviour of the upper branch of the neutral curve

For flows with an inflexion point, Lin (1945, p. 282) has shown that along the upper branch of the neutral curve $R \sim \text{const.}\,(\alpha-\alpha_s)^{-2}$ as $\alpha \to \alpha_s$. In the derivation of this result, however, it was assumed that α_s and c_s were both small. The value of the constant appearing in this result is thus only an approximate one and, in fact, it formally vanishes if $U'(0) = 0$. To avoid these difficulties, it is necessary to obtain the precise behaviour of $\Phi(0)/\Phi'(0)$ to at least first order in $\alpha-\alpha_s$ and $c-c_s$ as $\alpha \to \alpha_s$ and $c \to c_s$. This could easily be done by a slight modification of the discussion given in I, §4 and the results so obtained would remain valid whether $U'(0)$ vanished or not. It is possible to do rather more, however, with no additional complications; namely, to obtain the behaviour of $\Phi(y)$ itself to first order in $\alpha-\alpha_s$ and $c-c_s$.

For this purpose it is necessary to consider a second solution, $\Psi_s(y)$ say, of the limiting inviscid equation. A standard form of this solution can conveniently be defined by

$$\Psi_s(y) = \Phi_s(y)\int_{y_s}^{y} \{\Phi_s(y)\}^{-2}\,dy, \tag{4.3}$$

and a few of its properties may be briefly noted:

$$W(\Phi_s, \Psi_s) = 1, \tag{4.4}$$

$$\Psi_s(0) = -1/\Phi_s'(0), \quad \Psi_s(y_s) = 0, \quad \Psi_s'(y_s) = 1/\Phi_s(y_s), \tag{4.5}$$

and

$$\Psi_s(y) \sim e^{\alpha_s y}/2\alpha_s K \quad \text{as} \quad y \to \infty. \tag{4.6}$$

Consider now an expansion of $\Phi(y)$ in powers of both $\alpha-\alpha_s$ and $c-c_s$ of the form

$$\Phi(y) = \Phi_s(y) + \Phi_1(y)\,(\alpha-\alpha_s) + \Phi_2(y)\,(c-c_s) + \dots, \tag{4.7}$$

where Φ_1 and Φ_2 must then satisfy the equations

$$(U-c_s)\,(\Phi_1''-\alpha_s^2\,\Phi_1) - U''\Phi_1 = 2\alpha_s(U-c_s)\,\Phi_s \tag{4.8}$$

and

$$(U-c_s)\,(\Phi_2''-\alpha_s^2\,\Phi_2) - U''\Phi_2 = U''(U-c_s)^{-1}\,\Phi_s. \tag{4.9}$$

These equations must be solved subject to the conditions that Φ_1 and Φ_2 remain bounded as $y \to \infty$ and, to conform to our normalization convention, that Φ_1

and Φ_2 vanish at y_c. To the present order of approximation, this last condition is equivalent to the requirement that Φ_1 and Φ_2 vanish at y_s. Thus, we have

$$\Phi_1 = -2\alpha_s\left\{\Psi_s\int_y^\infty \Phi_s^2 dy + \Phi_s\int_{y_s}^y \Phi_s\Psi_s dy\right\} \tag{4.10}$$

and
$$\Phi_2 = -\Psi_s\int_y^\infty \frac{U''}{(U-c_s)^2}\,\Phi_s^2 dy - \Phi_s\int_{y_s}^y \frac{U''}{(U-c_s)^2}\,\Phi_s\Psi_s dy. \tag{4.11}$$

In equation (4.11) the path of integration for the first integral must lie below the critical point y_s; the integrand of the second integral, however, is regular at y_s. At $y = 0$, Φ_1 and Φ_2 have the values

$$\Phi_1(0) = \frac{2\alpha_s}{\Phi_s'(0)}\int_0^\infty \Phi_s^2 dy \quad\text{and}\quad \Phi_2(0) = \frac{1}{\Phi_s'(0)}\int_0^\infty \frac{U''}{(U-c_s)^2}\,\Phi_s^2 dy \tag{4.12}$$

and are thus independent of Ψ_s. It may be noticed that $\Phi_1(0)$ is real but that $\Phi_2(0)$ is complex with real and imaginary parts given by

$$\left.\begin{aligned}\Phi_{2r}(0) &= \frac{1}{\Phi_s'(0)}\,\mathscr{P}\int_0^\infty \frac{U''}{(U-c_s)^2}\,\Phi_s^2 dy\\ \text{and}\qquad \Phi_{2i}(0) &= \pi\frac{U'''(y_s)}{\{U'(y_s)\}^2}\,\frac{\Phi_s^2(y_s)}{\Phi_s'(0)},\end{aligned}\right\} \tag{4.13}$$

where $\mathscr{P}$ denotes the principal value of the integral.

These results can be used not only to obtain the asymptotic behaviour of the upper branch of the neutral curve as $R\to\infty$ but also to demonstrate the existence, near the neutral mode Φ_s, of a neighbouring unstable solution. For this latter purpose we consider the inviscid form of the characteristic equation (3.2) and immediately find that

$$c - c_s \to -\frac{\Phi_1(0)\,\Phi_2^*(0)}{|\Phi_2(0)|^2}(\alpha-\alpha_s) \quad\text{as}\quad c\to c_s \quad\text{and}\quad \alpha\to\alpha_s. \tag{4.14}$$

This result is clearly equivalent to Lin's formula (1955, p. 123) for $(\partial c/\partial\alpha^2)_{\alpha=\alpha_s}$. In particular, the imaginary part of equation (4.14) is

$$c_i \to \frac{\Phi_1(0)\,\Phi_{2i}(0)}{|\Phi_2(0)|^2}(\alpha-\alpha_s), \tag{4.15}$$

where the sign of the coefficient in this expression is determined by the sign of $U'''(y_s)$. If $U'''(y_s) < 0$, as it is in the present problem, then $c_i \gtrless 0$ when $\alpha \lessgtr \alpha_s$.

To obtain the asymptotic behaviour of the upper branch of the neutral curve we must consider the asymptotic form of the characteristic equation (3·2) as $R\to\infty$ with $c_i = 0$

$$-\frac{1}{y_s\Phi_s'(0)}\{\Phi_1(0)(\alpha-\alpha_s)+\Phi_2(0)(c-c_s)+\ldots\} \sim \frac{e^{\frac{1}{4}\pi i}}{z^{\frac{3}{2}}}, \tag{4.16}$$

where
$$z = (\alpha R U_c')^{\frac{1}{3}}y_c \to (\alpha_s R U_s')^{\frac{1}{3}}y_s\{1+O(\alpha-\alpha_s, c-c_s)\}. \tag{4.17}$$

By eliminating z between the real and imaginary parts of equation (4.16), we obtain

$$c - c_s \to \frac{\Phi_1(0)}{\Phi_{2i}(0)-\Phi_{2r}(0)}(\alpha-\alpha_s) \tag{4.18}$$

and the imaginary part itself then gives

$$R \sim \frac{1}{2\pi^2 \alpha_s y_s} \frac{U_s'^3}{U_s'''^2} \left\{\frac{\Phi_s'(0)}{\Phi_s(y_s)}\right\}^4 (c-c_s)^{-2}. \tag{4.19}$$

These results remain valid whether $U'(0)$ vanishes or not, but they are not valid if $U'''(y_s)$ vanishes. For a discussion of the purely inviscid problems when $U'''(y_s) = 0$, see Lin (1945, p. 224).

For the particular velocity distribution (1.1), we obtain the following numerical values for the integrals appearing in equations (4.12) and (4.13)

$$\int_0^\infty \Phi_s^2 dy = 0{\cdot}88360 \quad \text{and} \quad \mathscr{P}\int_0^\infty \frac{U''}{(U-c_s)^2} \Phi_s^2 dy = -5{\cdot}9895. \tag{4.20}$$

From these results we then have

$$\left.\begin{aligned} c-c_s &\to 0{\cdot}4963\,(\alpha_{1s}-\alpha_1) \\ \text{and} \qquad R_1 &\sim 0{\cdot}07602\,(c-c_s)^{-2} \quad \text{or} \quad R_1 \sim 0{\cdot}3087\,(\alpha_{1s}-\alpha_1)^{-2}. \end{aligned}\right\} \tag{4.21}$$

The asymptotic behaviour of the lower branch of the neutral curve

Along the lower branch of the neutral curve, $\alpha \to 0$ and we can therefore use the method of approximation described in I, §4. In terms of $\Omega(y)$, defined by equation (4.4) of I, the characteristic equation (3.2) becomes

$$(c^2/y_c)\,\Omega(0) = F(z), \tag{4.22}$$

where

$$\Omega(0) = 1/\alpha(1-c)^2 + \Omega_0(0) + \Omega_1(0)\,\alpha + \ldots. \tag{4.23}$$

For the present purposes it is sufficient to consider only the one term

$$\Omega_0(0) = -\frac{1}{(1-c)^2}\int_0^\infty \left\{\left(\frac{U-c}{1-c}\right)^2 - \left(\frac{1-c}{U-c}\right)^2\right\} dy, \tag{4.24}$$

where the path of integration must lie below the critical point. The imaginary part of $\Omega_0(0)$ is given by the usual expression

$$\Omega_{0i}(0) = -\pi U_c''/U_c'^3, \tag{4.25}$$

but the real part of $\Omega_0(0)$ cannot, in general, be evaluated explicitly. The behaviour of $\Omega_{0r}(0)$ for small values of c, however, depends very critically on whether $U'(0)$ vanishes or not. If $U'(0) = 0$, then it can be shown that $\Omega_{0r}(0) = O(1)$ as $c \to 0$ and this is sufficient for the present purposes. From equation (4.25) we then have

$$\Omega_0(0) = -\frac{\pi i}{2^{\frac{3}{2}} U_0''^{\frac{1}{2}}} c^{-\frac{3}{2}} + O(c^{-1}) \quad \text{as} \quad c \to 0. \tag{4.26}$$

The imaginary part of equation (4.22) then shows that

$$F_i(z) \to -\tfrac{1}{4}\pi \quad \text{as} \quad c \to 0 \tag{4.27}$$

and this condition fixes the point in figure 1 corresponding to the lower asymptote. From Miles's (1960) tables of the Tietjens function, we find that the value of z for which equation (4.27) is satisfied is $z_* = 0{\cdot}488$ and that $F_r(z_*) = 1{\cdot}580$. From the real part of equation (4.22) we have

$$c^{\frac{3}{2}} \to (2/U_0'')^{\frac{1}{2}} F_r(z_*)\,\alpha \tag{4.28}$$

and the asymptote to the lower branch of the neutral curve is then given by

$$R^{\frac{1}{3}} \sim \frac{U_0''^{\frac{5}{9}}}{2^{\frac{8}{9}}} \frac{z_*}{\{F_r(z_*)\}^{\frac{4}{9}}} \alpha^{-\frac{7}{9}}. \tag{4.29}$$

The power of α appearing in equation (4.29) differs from the usual boundary-layer results and can be traced to the vanishing of $U'(0)$. For the particular velocity distribution (1.1) we have

$$c^{\frac{3}{2}} \to 1{\cdot}61\,\alpha_1 \quad \text{and} \quad R_1^{\frac{1}{3}} \sim 0{\cdot}309\alpha_1^{-\frac{7}{9}}. \tag{4.30}$$

5. Concluding remarks

In spite of the rather special nature of the velocity profile considered in this paper, it would appear that the stability characteristics of a boundary layer at separation have been sufficiently well approximated to warrant a few general conclusions. On the basis of the usual asymptotic theory, it can be concluded that such a flow has a non-zero minimum critical Reynolds number and that the value of $R_{\min}$, though comparatively small, is not so small as to seriously violate the basic assumptions of the theory. The curve of neutral stability has the general form that one would expect of a flow having an inflexion point, and only the asymptote to its lower branch depends in any essential way on the fact that $U'(0) = 0$.

One minor defect in the present analysis, however, should be briefly mentioned. This concerns the asymptotic behaviour of the lower branch of the neutral curve which was obtained in §4 from the characteristic equation (3.2). As $c \to 0$, however, there are two critical points at $\pm(2c/U_0'')^{\frac{1}{2}}$ which coalesce to form a single turning point of the second order and the viscous solution (3.1) is no longer valid in this limiting situation. To obtain an asymptotic approximation to $\phi_3(y)$ that is valid in a domain containing both critical points and remains valid as $c \to 0$ it would be necessary to use a comparison equation of the Weber rather than the Airy type. Furthermore, since $c \to 0$ like $(\alpha R)^{-\frac{1}{2}}$, the relevant Weber equation can be transformed to a parameter-free form, thus precluding further simplification in terms of either Airy or exponential functions. Although such refinements in the analysis are of some mathematical interest, it seems unlikely that they would make more than slight quantitative differences to the present results.

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